Chapter 8

Determinants and Matrices

Contents

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The theory of determinants may be said to have begun with G.W. Leibnitz (1646-1716) *who gave a rule for the solution of simultaneous linear equations equivalent to that of the Chinese. This rule was simplified by* G. Cramer (1704-1752), *a Swiss mathematician, in* 1750. E. Bezout (1730-1783), *a French mathematician simplified it further in* 1764. *However, A.T. Vandermonde (1735-1796) gave the first systematic account of determinants in* 1771. P.S. Laplace (1749-1827), *in* 1772, *gave the general method of expanding a determinant in terms of minors. In* 1773, J.L,. Lagrange (1736-1813) *treated determinants fo order 2 and 3 and used them for purposes other than the solution of equations.* Carl F. Gauss (1777-1855) *used determinants in the theory of numbers.*

J.J. Sylvester, *who was very fond of assigning imaginative names to his creations and inventions, wrote down in 1850, certain terms and expressions in the form of a rectangular arrangement, Sylvester gave this rectangular arrangement the name 'matrix'. In particular, the names of* Sir William Rowan Hamilton, (1805-1865) *and Arthur Cayley deserve special mention. Sir Hamilton, in* 1853, *and Arthur Cayley, in 1858, made significant contributions to the theory of matrices.*

8.1 Determinants

8.1.1 Definition

(1) Consider two equations, $a_1x + b_1y = 0$ .....(i) and $a_2x + b_2y = 0$ .....(ii)

Multiplying (i) by b_2 and (ii) by b_1 and subtracting, dividing by *x*, we get, $a_1b_2 - a_2b_1 = 0$

The result $a_1b_2 - a_2b_1$ is represented by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Which is known as determinant of order two and $a_1b_2 - a_2b_1$ is the expansion of this determinant. The horizontal lines are called rows and vertical lines are called columns.

Now let us consider three homogeneous linear equations

$$a_1x + b_1y + c_1z = 0,\ a_2x + b_2y + c_2z = 0 \text{ and } a_3x + b_3y + c_3z = 0$$

Eliminated *x, y, z* from above three equations we obtain

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = 0 \quad .....(iii)$$

The L.H.S. of (iii) is represented by $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Its contains three rows and three columns, it is called a determinant of third order.

Note : ❑ The number of elements in a second order is $2^2 = 4$ and the number of elements in a third order determinant is $3^2 = 9$.

(2) **Rows and columns of a determinant :** In a determinant horizontal lines counting from top 1st, 2nd, 3rd,..... respectively known as rows and denoted by $R_1, R_2, R_3, ...$ and vertical lines counting left to right, 1st, 2nd, 3rd,..... respectively known as columns and denoted by $C_1, C_2, C_3,.....$

(3) **Shape and constituents of a determinant :** Shape of every determinant is square. If a determinant of *n* order then it contains *n* rows and *n* columns.

i.e., Number of constituents in determinants = n^2

(4) **Sign system for expansion of determinant :** Sign system for order 2, order 3, order 4,..... are given by $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$, $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$, $\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$,.....

8.1.2 Expansion of Determinants

Unlike a matrix, determinant is not just a table of numerical data but (quite differently) a short hand way of writing algebraic expression, whose value can be computed when the values of terms or elements are known.

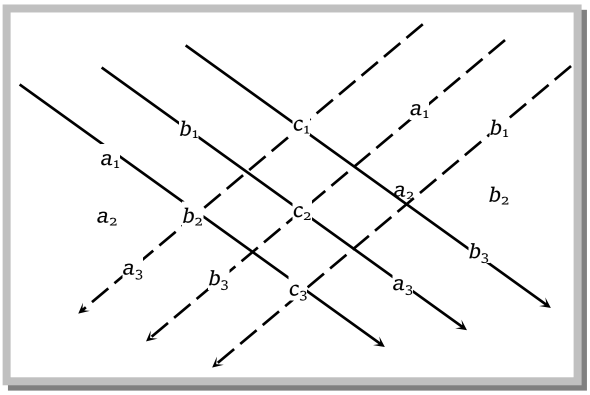
(1) The 4 numbers a_1, b_1, a_2, b_2 arranged as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant of second order. These numbers are called elements of the determinant. The value of the determinant is defined as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

The expanded form of determinant has 2! terms.

(2) The 9 numbers $a_r, b_r, c_r \ (r = 1, 2, 3)$ arranged as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a determinant of third order. Take any row (or column); the value of the determinant is the sum of products of the elements of the row (or column) and the corresponding determinant obtained by omitting the row and the column of the element with a proper sign, given by the rule $(-1)^{i+j}$, where i and j are the number of rows and the number of columns respectively of the element of the row (or the column) chosen. Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

The diagonal through the left-hand top corner which contains the element a_1, b_2, c_3 is called the leading diagonal or principal diagonal and the terms are called the leading terms. The expanded form of determinant has 3! terms.

Short cut method or Sarrus diagram method : To find the value of third order determinant, following method is also useful



Taking product of R.H.S. diagonal elements positive and L.H.S. diagonal elements negative and adding them. We get the value of determinant as $= a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$

Note : ❑ This method does not work for determinants of order greater than three.

8.1.3 Evaluation of Determinants

If A is a square matrix of order 2, then its determinant can be easily found. But to evaluate determinants of square matrices of higher orders, we should always try to introduce zeros at maximum number of places in a particular row (column) by using the properties and then we should expand the determinant along that row (column).

We shall be using the following notations to evaluate a determinant :

(1) R_i to denote i^{th} row.

(2) $R_i \leftrightarrow R_j$ to denote the interchange of i^{th} and j^{th} rows.

(3) $R_i \to R_i + \lambda R_j$ to denote the addition of λ times the elements of j^{th} row to the corresponding elements of i^{th} row.

(4) $R_i(\lambda)$ to denote the multiplication of all element of i^{th} row by λ.

Similar notations are used to denote column operations if R is replaced by C.

8.1.4 Properties of Determinants

P-1 : The value of determinant remains unchanged, if the rows and the columns are interchanged.

If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$. Then $D' = D$, D and D' are transpose of each other.

Note : ❑ Since the determinant remains unchanged when rows and columns are interchanged, it is obvious that any theorem which is true for 'rows' must also be true for 'columns'.

P-2 : If any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value but is changed in sign only.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = -D$

P-3 : If a determinant has two rows (or columns) identical, then its value is zero.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$. Then, $D = 0$

P-4 : If all the elements of any row (or column) be multiplied by the same number, then the value of determinant is multiplied by that number.

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = kD$

P-5 : If each element of any row (or column) can be expressed as a sum of two terms, then the determinant can be expressed as the sum of the determinants.

e.g., $\begin{vmatrix} a_1+x & b_1+y & c_1+z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

P-6 : The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column)

e.g., $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} a_1+ma_2 & b_1+mb_2 & c_1+mc_2 \\ a_2 & b_2 & c_2 \\ a_3-na_1 & b_3-nb_1 & c_3-nc_1 \end{vmatrix}$. Then $D' = D$

Note : ❑ It should be noted that while applying **P-6** at least one row (or column) must remain unchanged.

P-7 : If all elements below leading diagonal or above leading diagonal or except leading diagonal elements are zero then the value of the determinant equal to multiplied of all leading diagonal elements.

e.g., $\begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & 0 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix} = a_1b_2c_3$

P-8 : If a determinant D becomes zero on putting $x=\alpha$, then we say that $(x-\alpha)$ is factor of determinant.

e.g., if $D = \begin{vmatrix} x & 5 & 2 \\ x^2 & 9 & 4 \\ x^3 & 16 & 8 \end{vmatrix}$. At $x=2$, $D=0$ (because C_1 and C_2 are identical at $x=2$)

Hence $(x-2)$ is a factor of D.

Note : ❑ It should be noted that while applying operations on determinants then at least one row (or column) must remain unchanged.or, Maximum number of operations = order or determinant –1

❑ It should be noted that if the row (or column) which is changed by multiplied a non zero number, then the determinant will be divided by that number.

Example: 1 If $n \neq 3k$ and 1, ω, ω^2 are the cube roots of unity, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ has the value

[Pb. CET 1991; Rajasthan PET 2001; AIEEE 2003]

(a) 0 (b) ω (c) ω^2 (d) 1

Solution: (a) Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 1+\omega^n+\omega^{2n} & \omega^n & \omega^{2n} \\ 1+\omega^n+\omega^{2n} & 1 & \omega^n \\ 1+\omega^n+\omega^{2n} & \omega^{2n} & 1 \end{vmatrix} = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & 1 & \omega^n \\ 0 & \omega^{2n} & 1 \end{vmatrix} = 0 \qquad (\because 1+\omega^n+\omega^{2n} = 0 \text{ if } n \text{ is not multiple of 3})$$

Example: 2 $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} =$ **[Rajasthan PET 1992; Kerala (Engg.) 2002]**

(a) $xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$ (b) xyz (c) $1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ (d) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

Solution: (a) $\Delta = xyz\begin{vmatrix} 1+\frac{1}{x} & \frac{1}{x} & \frac{1}{x} \\ \frac{1}{y} & 1+\frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & 1+\frac{1}{z} \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{y} & 1+\frac{1}{y} & \frac{1}{y} \\ \frac{1}{z} & \frac{1}{z} & 1+\frac{1}{z} \end{vmatrix}$ (by $R_1 \to R_1 + R_2 + R_3$)

$= xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{y} & 1 & 0 \\ \frac{1}{z} & 0 & 1 \end{vmatrix}$ (by $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$)

$= xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

Trick : Put $x=1,\ y=2$ and $z=3$, then $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 2(11)-1(3)+1(1-3)=17$.

option (a) gives $1\times 2\times 3\left(1+\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\right)=17$

Example: 3 The value of $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$ is equal to zero, where m is

(a) 6 (b) 4 (c) 5 (d) None of these

Solution: (c) $\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$

Applying $C_2 \to C_1 + C_2$

$\Delta = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_4 + {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_6 + {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_8 + {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0 \Rightarrow \Delta = \begin{vmatrix} {}^{10}C_4 & {}^{11}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{12}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{13}C_9 & {}^{13}C_{m+4} \end{vmatrix} = 0$

Clearly $m=5$ satisfies the above result [$\because C_2, C_3$ will be identical]

Example: 4 If $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ are in $G.P.$ then the value of the determinant $\begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$ is

[AIEEE 2004; IIT 1993]

(a) –2 (b) 1 (c) 2 (d) 0

Solution: (d) $\because a_1, a_2, a_3, \ldots\ldots, a_n, \ldots\ldots$ are in $G.P.$

$\therefore a_{n+1}^2 = a_n . a_{n+2} \Rightarrow 2\log a_{n+1} = \log a_n + \log a_{n+2}$

$a_{n+4}^2 = a_{n+3}.a_{n+5} \Rightarrow 2\log a_{n+4} = \log a_{n+3} + \log a_{n+5}$

$a_{n+7}^2 = a_{n+6}.a_{n+8} \Rightarrow 2\log a_{n+7} = \log a_{n+6} + \log a_{n+8}$

Putting these values in the second column of the given determinant, we get

$$\Delta = \frac{1}{2}\begin{vmatrix} \log a_n & \log a_n + \log a_{n+2} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+3} + \log a_{n+5} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+6} + \log a_{n+8} & \log a_{n+8} \end{vmatrix}$$

$=\frac{1}{2}(0)=0$ [$\because C_2$ is the sum of two elements, first identical with C_1 and second with C_3]

Example: 5 The value of $\begin{vmatrix} 1 & 1 & 1 \\ (2^x+2^{-x})^2 & (3^x+3^{-x})^2 & (5^x+5^{-x})^2 \\ (2^x-2^{-x})^2 & (3^x-3^{-x})^2 & (5^x-5^{-x})^2 \end{vmatrix}$

(a) 0 (b) 30^x (c) 30^{-x} (d) None of these

Solution: (a) Applying $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2.2^x.2.2^{-x} & 2.3^x.2.3^{-x} & 2.5^x.2.5^{-x} \\ (2^x-2^{-x})^2 & (3^x-3^{-x})^2 & (5^x-5^{-x})^2 \end{vmatrix} = 4\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ (2^x-2^{-x})^2 & (3^x-3^{-x})^2 & (5^x-5^{-x})^2 \end{vmatrix} = 0 \; [\because R_1 \text{ and } R_2 \text{ are identical}]$$

Trick : Putting $x=0$, we get option (a) is correct

Example: 6 If *x, y, z* are integers in *A.P.* lying between 1 and 9 and *x*51, *y*41 and *z*31 are three digit numbers then the value of $\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x+y+z$ (b) $x-y+z$ (c) 0 (d) None of these

Solution: (c) $\because \; x51 = 100x+50+1$,

$y41 = 100y+40+1$

$z31 = 100z+30+1$

$$\therefore \; \Delta = \begin{vmatrix} 5 & 4 & 3 \\ 100x+50+1 & 100y+40+1 & 100z+30+1 \\ x & y & z \end{vmatrix}$$

Applying $R_2 \to R_2 - 100R_3 - 10R_1$

$$\Delta = \begin{vmatrix} 5 & 4 & 3 \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = x-2y+z$$

$\because$ *x, y, z* are in A.P. , $\therefore \; x-2y+z=0$, $\therefore \; \Delta = 0$

Example: 7 If $a \ne b \ne c$, the value of *x* which satisfies the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0$ is

[EAMCET 1988; Karnataka CET 1991; MNR 1980; MP PET 1988, 99, 2001; DCE 2001]

(a) $x=0$ (b) $x=a$ (c) $x=b$ (d) $x=c$

Solution: (a) Expanding determinant, we get, $\Delta = -(x-a)[-(x+b)(x-c)]+(x+b)[(x+a)(x+c)]=0 \Rightarrow 2x^3-(2\,\Sigma ab)x=0$

$\Rightarrow$ Either $x=0$ or $x^2=\Sigma ab$. Since $x=0$ satisfies the given equation.

Trick : On putting $x=0$, we observe that the determinant becomes $\Delta_{x=0} = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

$\therefore \; x=0$ is a root of the given equation.

Example: 8 The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0 (b) 2 (c) 1 (d) 3

Solution: (c) $(2\cos x+\sin x)\begin{vmatrix}1 & \cos x & \cos x\\ 1 & \sin x & \cos x\\ 1 & \cos x & \sin x\end{vmatrix}=0$

Applying, $R_2\to R_2-R_1$ and $R_3\to R_3-R_1$

$$(2\cos x+\sin x)\begin{vmatrix}1 & \cos x & \cos x\\ 0 & \sin x-\cos x & 0\\ 0 & 0 & \sin x-\cos x\end{vmatrix}=0 \Rightarrow (2\cos x+\sin x)(\sin x-\cos x)^2=0$$

$\therefore \tan x=-2,1$ But $\tan x\neq -2$ in $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$. Hence $\tan x=1\Rightarrow x=\frac{\pi}{4}$

Example: 9 If $p\lambda^4+q\lambda^3+r\lambda^2+s\lambda+t=\begin{vmatrix}\lambda^2+3\lambda & \lambda-1 & \lambda+3\\ \lambda+1 & 2-\lambda & \lambda-4\\ \lambda-3 & \lambda+4 & 3\lambda\end{vmatrix}$, then value of t is **[IIT 1981]**

(a) 16 (b) 18 (c) 17 (d) 19

Solution: (b) Since it is an identity in λ so satisfied by every value of λ. Now put $\lambda=0$ in the given equation, we have

$$t=\begin{vmatrix}0 & -1 & 3\\ 1 & 2 & -4\\ -3 & 4 & 0\end{vmatrix}=-12+30=18$$

Example: 10 If $f(x)=\begin{vmatrix}1 & x & (x+1)\\ 2x & x(x-1) & (x+1)x\\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1)\end{vmatrix}$, then $f(100)$ is equal to **[IIT 1999, MP PET 2000]**

(a) 0 (b) 1 (c) 100 (d) –100

Solution: (a) $f(x)=\begin{vmatrix}1 & x & (x+1)\\ 2x & x(x-1) & (x+1)x\\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1)\end{vmatrix}$

Applying $C_3\to C_3-C_2$, we get

$$f(x)=\begin{vmatrix}1 & x & 1\\ 2x & x(x-1) & 2x\\ 3x(x-1) & x(x-1)(x-2) & 3x(x-1)\end{vmatrix}=0$$. Hence $f(100)=0$

Example: 11 The value of $\begin{vmatrix}1 & 0 & 0 & 0 & 0\\ 2 & 2 & 0 & 0 & 0\\ 4 & 4 & 3 & 0 & 0\\ 5 & 5 & 5 & 4 & 0\\ 6 & 6 & 6 & 6 & 5\end{vmatrix}$ is

(a) 6! (b) 5! (c) $1.2^2.3.4^3.5^4.6^4$ (d) None of these

Solution: (b) The elements in the leading diagonal are 1, 2, 3, 4, 5. On one side of the leading diagonal all the elements are zero.

$\therefore$ The value of the determinant

= The product of the elements in the leading diagonal = 1.2.3.4.5 = 5!

Example: 12 The determinant $\begin{vmatrix}a & b & a\alpha+b\\ b & c & b\alpha+c\\ a\alpha+b & b\alpha+c & 0\end{vmatrix}=0$ if **[DCE 2000, 2001]**

(a) *a, b, c* are in A.P.

(b) *a, b, c* are in G.P. or $(x-\alpha)$ is a factor of $ax^2+2bx+c=0$

(c) *a, b, c* are in H.P.

(d) α is a root of the equation

Solution: (b) Applying $R_3 \to R_3 - \alpha R_1 - R_2$, we get $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ 0 & 0 & -a\alpha^2 - b\alpha - b\alpha - c \end{vmatrix} = 0$

$\Rightarrow -(a\alpha^2 + 2b\alpha + c)(ac - b^2) = 0 \Rightarrow a\alpha^2 + 2b\alpha + c = 0$ or $b^2 = ac$

$\Rightarrow$ $x = \alpha$ is a root of $ax^2 + 2bx + c = 0$ or a, b, c are in G.P.

$\Rightarrow$ $(x - \alpha)$ is a factor of $ax^2 + 2bx + c = 0$ or a, b, c are in G.P.

8.1.5 Minors and Cofactors

(1) **Minor of an element :** If we take the element of the determinant and delete (remove) the row and column containing that element, the determinant left is called the minor of that element. It is denoted by M_{ij}

Consider the determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of minors

$M = \begin{vmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{vmatrix}$,

where M_{11} = minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, M_{12} = minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

M_{13} = minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Similarly, we can find the minors of other elements . Using this concept the value of determinant can be

$$\Delta = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

or, $\Delta = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$ or, $\Delta = a_{31}M_{31} - a_{32}M_{32} + a_{33}M_{33}$.

(2) **Cofactor of an element :** The cofactor of an element a_{ij} (*i.e.* the element in the i^{th} row and j^{th} column) is defined as $(-1)^{i+j}$ times the minor of that element. It is denoted by C_{ij} or A_{ij} or F_{ij}.

$C_{ij} = (-1)^{i+j} M_{ij}$

If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then determinant of cofactors is $C = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix}$, where

$C_{11} = (-1)^{1+1} M_{11} = +M_{11}$, $C_{12} = (-1)^{1+2} M_{12} = -M_{12}$ and $C_{13} = (-1)^{1+3} M_{13} = +M_{13}$

Similarly, we can find the cofactors of other elements.

Note : ❑ The sum of products of the element of any row with their corresponding cofactor is equal to the value of determinant *i.e.* $\Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ $= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$

where the capital letters C_{11}, C_{12}, C_{13} *etc.* denote the cofactors of a_{11}, a_{12}, a_{13} *etc.*

- In general, it should be noted

  $a_{i1}C_{j1} + a_{i2}C_{j2} + a_{i3}C_{j3} = 0$, if $i \neq j$ or $a_{1i}C_{1j} + a_{2i}C_{2j} + a_{3i}C_{3j} = 0$, if $i \neq j$

- If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding cofactors, then if $\Delta = 0$, then $\Delta^C = 0$, $\Delta' = \Delta^{n-1}$, where n is the order of the determinant.

Example: 13 The cofactor of the element 4 in the determinant $\begin{vmatrix} 1 & 3 & 5 & 1 \\ 2 & 3 & 4 & 2 \\ 8 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \end{vmatrix}$ is **[MP PET 1987]**

(a) 4 (b) 10 (c) –10 (d) –4

Solution: (b) The cofactor of element 4, in the 2nd row and 3rd column is $(-1)^{2+3}\begin{vmatrix} 1 & 3 & 1 \\ 8 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$ =

$-\{1(-2) - 3(8-0) + 1.16\} = 10$

Example: 14 If $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and A_1, B_1, C_1 denote the cofactors of a_1, b_1, c_1 respectively, then the value of the determinant $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$ is

(a) Δ (b) Δ^2 (c) Δ^3 (d) 0

Solution: (b) We know that $\Delta.\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} . \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} \Sigma a_1A_1 & 0 & 0 \\ 0 & \Sigma a_2A_2 & 0 \\ 0 & 0 & \Sigma a_3A_3 \end{vmatrix} = \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$

$\Rightarrow \Delta' = \Delta^2$

Trick : According to property of cofactors $\Delta' = \Delta^{n-1} = \Delta^2$ ($\because$ Hence $n = 3$)

Example: 15 If the value of a third order determinant is 11, then the value of the square of the determinant formed by the cofactors will be

(a) 11 (b) 121 (c) 1331 (d) 14641

Solution: (d) $\Delta' = \Delta^{n-1} = \Delta^{3-1} = \Delta^2 = (11)^2 = 121$. But we have to find the value of the square of the determinant, so required value is $(121)^2 = 14641$.

8.1.6 Product of two Determinants

Let the two determinants of third order be,

$D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$. Let D be their product.

(1) **Method of multiplying (Row by row) :** Take the first row of D_1 and the first row of D_2 *i.e.* a_1, b_1, c_1 and $\alpha_1, \beta_1, \gamma_1$ multiplying the corresponding elements and add. The result is $a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1$ is the first element of first row of D.

Now similar product first row of D_1 and second row of D_2 gives $a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2$ is the second element of first row of D, and the product of first row D_1 and third row of D_2 gives $a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3$ is the third element of first row of D. The second row and third row of D is obtained by multiplying second row and third row of D_1 with 1^{st}, 2^{nd}, 3^{rd} row of D_2, in the above manner.

Hence,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Note : ❑ We can also multiply rows by columns or columns by rows or columns by columns.

Example: 16 For all values of A, B, C and P, Q, R the value of $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$ is

(a) 0 (b) $\cos A \cos B \cos C$ (c) $\sin A \sin B \sin C$ (d) $\cos P \cos Q \cos R$

Solution: (a) The determinant can be expanded as

$$\begin{vmatrix} \cos A\cos P + \sin A\sin P & \cos A\cos Q + \sin A\sin Q & \cos A\cos R + \sin A\sin R \\ \cos B\cos P + \sin B\sin P & \cos B\cos Q + \sin B\sin Q & \cos B\cos R + \sin B\sin R \\ \cos C\cos P + \sin C\sin P & \cos C\cos Q + \sin C\sin Q & \cos C\cos R + \sin C\sin R \end{vmatrix} = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix} = 0$$

Example: 17 $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix} =$ **[Tamilnadu (Engg.) 2002]**

(a) 7 (b) 10 (c) 13 (d) 17

Solution: (b) $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix} = \left(\frac{\log 512}{\log 3} \times \frac{\log 9}{\log 4} - \frac{\log 3}{\log 4} \times \frac{\log 8}{\log 3}\right) \times \left(\frac{\log 3}{\log 2} \times \frac{\log 4}{\log 3} - \frac{\log 3}{\log 8} \times \frac{\log 4}{\log 3}\right)$

$$= \left(\frac{\log 2^9}{\log 3} \times \frac{\log 3^2}{\log 2^2} - \frac{\log 2^3}{\log 2^2}\right) \times \left(\frac{\log 2^2}{\log 2} - \frac{\log 2^2}{\log 2^3}\right) = \left(\frac{9 \times 2}{2} - \frac{3}{2}\right)\left(2 - \frac{2}{3}\right) = 10$$

8.1.7 Summation of Determinants

Let $\Delta_r = \begin{vmatrix} f(r) & a & l \\ g(r) & b & m \\ h(r) & c & n \end{vmatrix}$, where a, b, c, l, m and n are constants, independent of r.

Then, $\sum_{r=1}^{n}\Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & a & l \\ \sum_{r=1}^{n} g(r) & b & m \\ \sum_{r=1}^{n} h(r) & c & n \end{vmatrix}$. Here function of r can be the elements of only one row or one column.

Example: 18 If $D_p = \begin{vmatrix} p & 15 & 8 \\ p^2 & 35 & 9 \\ p^3 & 25 & 10 \end{vmatrix}$, then $D_1 + D_2 + D_3 + D_4 + D_5 =$

(a) 0 (b) 25 (c) 625 (d) None of these

Solution: (d) $\because D_1 = \begin{vmatrix} 1 & 15 & 8 \\ 1 & 35 & 9 \\ 1 & 25 & 10 \end{vmatrix}, D_2 = \begin{vmatrix} 2 & 15 & 8 \\ 4 & 35 & 9 \\ 8 & 25 & 10 \end{vmatrix}, D_3 = \begin{vmatrix} 3 & 15 & 8 \\ 9 & 35 & 9 \\ 27 & 25 & 10 \end{vmatrix}, D_4 = \begin{vmatrix} 4 & 15 & 8 \\ 16 & 35 & 9 \\ 64 & 25 & 10 \end{vmatrix}, D_5 = \begin{vmatrix} 5 & 15 & 8 \\ 25 & 35 & 9 \\ 125 & 25 & 10 \end{vmatrix}$

$\Rightarrow D_1 + D_2 + D_3 + D_4 + D_5 = \begin{vmatrix} 15 & 75 & 40 \\ 55 & 175 & 45 \\ 225 & 125 & 50 \end{vmatrix}$

$= 15(3125) - 75(-7375) + 40(-32500) = 46875 + 553125 - 1300000 = -700000$

Example: 19 The value of $\sum_{n=1}^{N} U_n$, if $U_n = \begin{vmatrix} n & 1 & 5 \\ n^2 & 2N+1 & 2N+1 \\ n^3 & 3N^2 & 3N \end{vmatrix}$ is

(a) 0 (b) 1 (c) –1 (d) None of these

Solution: (a) $\sum_{n=1}^{N} U_n = \begin{vmatrix} \frac{N(N+1)}{2} & 1 & 5 \\ \frac{N(N+1)(2N+1)}{6} & 2N+1 & 2N+1 \\ \left\{\frac{N(N+1)}{2}\right\}^2 & 3N^2 & 3N \end{vmatrix} = \frac{N(N+1)}{12}\begin{vmatrix} 6 & 1 & 5 \\ 4N+2 & 2N+1 & 2N+1 \\ 3N(N+1) & 3N^2 & 3N \end{vmatrix}$

Applying $C_3 \to C_3 + C_2$ $= \frac{N(N+1)}{12}\begin{vmatrix} 6 & 1 & 6 \\ 4N+2 & 2N+1 & 4N+2 \\ 3N(N+1) & 3N^2 & 3N(N+1) \end{vmatrix} = 0$ [$\because$ C_1 and C_3 are identical]

8.1.8 Differentiation and Integration of Determinants

(1) **Differentiation of a determinant :** (i) Let $\Delta(x)$ be a determinant of order two. If we write $\Delta(x) = |C_1 \quad C_2|$, where C_1 and C_2 denote the 1st and 2nd columns, then

$$\Delta'(x) = |C'_1 \quad C_2| + |C_1 \quad C'_2|$$

where C'_i denotes the column which contains the derivative of all the functions in the i^{th} column C_i.

In a similar fashion, if we write $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \end{vmatrix}$

(ii) Let $\Delta(x)$ be a determinant of order three. If we write $\Delta(x) = |C_1 \quad C_2 \quad C_3|$, then

$\Delta'(x) = |C'_1 \quad C_2 \quad C_3| + |C_1 \quad C'_2 \quad C_3| + |C_1 \quad C_2 \quad C'_3|$

and similarly if we consider $\Delta(x) = \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix}$, then $\Delta'(x) = \begin{vmatrix} R'_1 \\ R_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R'_2 \\ R_3 \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R'_3 \end{vmatrix}$

(iii) If only one row (or column) consists functions of x and other rows (or columns) are constant, *viz.*

Let $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$,

then $\Delta'(x) = \begin{vmatrix} f'_1(x) & f'_2(x) & f'_3(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and in general $\Delta^n(x) = \begin{vmatrix} f_1^n(x) & f_2^n(x) & f_3^n(x) \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

where n is any positive integer and $f^n(x)$ denotes the n^{th} derivative of $f(x)$.

Example: 20 If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then

[MNR 1986; Kurukshetra CEE 1998; UPSEAT 2000]

(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $\frac{d}{dx}(\Delta_1) = 3\Delta_2$ (c) $\frac{d}{dx}(\Delta_1) = 2(\Delta_2)^2$ (d) $\Delta_1 = 3\Delta_2^{3/2}$

Solution: (b) $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix} = x^3 - 3abx \Rightarrow \frac{d}{dx}(\Delta_1) = 3(x^2 - ab)$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix} = x^2 - ab$

$\therefore \frac{d}{dx}(\Delta_1) = 3(x^2 - ab) = 3\Delta_2$

Example: 21 If $y = \sin mx$, then the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$, where $y_n = \frac{d^n y}{dx^n}$ is

(a) m^9 (b) m^2 (c) m^3 (d) None of these

Solution: (d) $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = \begin{vmatrix} \sin mx & m\cos mx & -m^2\sin mx \\ -m^3\cos mx & m^4\sin mx & m^5\cos mx \\ -m^6\sin mx & -m^7\cos mx & m^8\sin mx \end{vmatrix}$

Taking $-m^6$ common from R_3, R_1 and R_3 becomes identical. Hence the value of determinant is zero.

(2) **Integration of a determinant :** Let $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ l & m & n \end{vmatrix}$, where a, b, c, l, m and n are constants.

$$\Rightarrow \int_a^b \Delta(x)dx = \begin{vmatrix} \int_a^b f(x)dx & \int_a^b g(x)dx & \int_a^b h(x)dx \\ a & b & c \\ l & m & n \end{vmatrix}$$

Note : ❑ If the elements of more than one column or rows are functions of x then the integration can be done only after evaluation/expansion of the determinant.

Example: 22 If $\Delta(x)=\begin{vmatrix}1 & \cos x & 1-\cos x\\ 1+\sin x & \cos x & 1+\sin x-\cos x\\ \sin x & \sin x & 1\end{vmatrix}$, then $\int_0^{\pi/2}\Delta(x)\,dx$ is equal to

(a) 1/4 (b) 1/2 (c) 0 (d) –1/2

Solution: (d) Applying $C_3 \to C_3 + C_2 - C_1$

$$\Delta(x)=\begin{vmatrix}1 & \cos x & 0\\ 1+\sin x & \cos x & 0\\ \sin x & \sin x & 1\end{vmatrix} = \cos x - \cos x(1+\sin x) = -\sin x\cos x$$

$$\therefore \int_0^{\pi/2}\Delta(x)dx = -\frac{1}{2}\int_0^{\pi/2}\sin 2x dx = -\frac{1}{2}\left[-\frac{\cos 2x}{2}\right]_0^{\pi/2} = \frac{1}{4}(\cos\pi - \cos 0) = -\frac{1}{2}$$

8.1.9 Application of Determinants in solving a system of Linear Equations

Consider a system of simultaneous linear equations is given by $\left.\begin{matrix}a_1x+b_1y+c_1z=d_1\\ a_2x+b_2y+c_2z=d_2\\ a_3x+b_3y+c_3z=d_3\end{matrix}\right\}$

.....(i)

A set of values of the variables *x, y, z* which simultaneously satisfy these three equations is called a solution. A system of linear equations may have a unique solution or many solutions, or no solution at all, if it has a solution (whether unique or not) the system is said to be consistent. If it has no solution, it is called an inconsistent system.

If $d_1 = d_2 = d_3 = 0$ in (i) then the system of equations is said to be a homogeneous system. Otherwise it is called a non-homogeneous system of equations.

Theorem 1 : (Cramer's rule) The solution of the system of simultaneous linear equations

$a_1x + b_1y = c_1$ .....(i) and $a_2x + b_2y = c_2$ .....(ii)

is given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}$, where $D = \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}$, $D_1 = \begin{vmatrix}c_1 & b_1\\ c_2 & b_2\end{vmatrix}$ and $D_2 = \begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}$, provided that $D \neq 0$

Note : ❑ Here $D = \begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}$ is the determinant of the coefficient matrix $\begin{bmatrix}a_1 & b_1\\ a_2 & b_2\end{bmatrix}$.

The determinant D_1 is obtained by replacing first column in D by the column of the right hand side

of the given equations. The determinant D_2 is obtained by replacing the second column in D by the right most column in the given system of equations.

(1) **Solution of system of linear equations in three variables by Cramer's rule :**

Theorem 2 : (Cramer's Rule) The solution of the system of linear equations

$a_1x + b_1y + c_1z = d_1$ .....(i)

$a_2x + b_2y + c_2z = d_2$ .....(ii)

$a_3x + b_3y + c_3z = d_3$ .....(iii)

is given by $x=\frac{D_1}{D}$, $y=\frac{D_2}{D}$ and $z=\frac{D_3}{D}$, where $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

$D_1=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $D_2=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, and $D_3=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, Provided that $D\neq 0$

Note :❑ Here D is the determinant of the coefficient matrix. The determinant D_1 is obtained by replacing the elements in first column of D by d_1, d_2, d_3. D_2 is obtained by replacing the element in the second column of D by d_1, d_2, d_3 and to obtain D_3, replace elements in the third column of D by d_1, d_2, d_3.

Theorem 3 : (Cramer's Rule) Let there be a system of n_simultaneous linear equation n unknown given by

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_n \end{aligned}$$

Let $D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{vmatrix}$ and let D_j, be the determinant obtained from D after replacing the j^{th} column by $\begin{vmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{vmatrix}$. Then, $x_1=\frac{D_1}{D}$, $x_2=\frac{D_2}{D}$,......, $x_n=\frac{D_n}{D}$, Provided that $D\neq 0$

(2) **Conditions for consistency**

Case 1 : For a system of 2 simultaneous linear equations with 2 unknowns

(i) If $D\neq 0$, then the given system of equations is consistent and has a unique solution given by $x=\frac{D_1}{D}, y=\frac{D_2}{D}$.

(ii) If $D=0$ and $D_1=D_2=0$, then the system is consistent and has infinitely many solutions.

(iii) If $D=0$ and one of D_1 and D_2 is non-zero, then the system is inconsistent.

Case 2 : For a system of 3 simultaneous linear equations in three unknowns

(i) If $D\neq 0$, then the given system of equations is consistent and has a unique solution given by $x=\frac{D_1}{D}$, $y=\frac{D_2}{D}$ and $z=\frac{D_3}{D}$

(ii) If $D=0$ and $D_1=D_2=D_3=0$, then the given system of equations is consistent with infinitely many solutions.

(iii) If $D=0$ and at least one of the determinants D_1, D_2, D_3 is non-zero, then given of equations is inconsistent.

(3) **Algorithm for solving a system of simultaneous linear equations by Cramer's rule (Determinant method)**

Step 1 : Obtain $D,\ D_1,\ D_2$ and D_3

Step 2 : Find the value of *D*. If $D \neq 0$, then the system of the equations is consistent has a unique solution. To find the solution, obtain the values of $D_1,\ D_2$ and D_3. The solutions is given by $x=\frac{D_1}{D},\ y=\frac{D_2}{D}$ and $z=\frac{D_3}{D}$. If $D=0$ go to step 3.

Step 3 : Find the values of D_1, D_2, D_3. If at least one of these determinants is non-zero, then the system is inconsistent. If $D_1=D_2=D_3=0$, then go to step 4

Step 4 : Take any two equations out of three given equations and shift one of the variables, say *z* on the right hand side to obtain two equations in *x, y*. Solve these two equations by Cramer's rule to obtain *x, y,* in terms of *z*.

Note: ❑ The system of following homogeneous equations $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$, $a_3x+b_3y+c_3z=0$ is always consistent.

If $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$, then this system has the unique solution $x=y=z=0$ known as **trivial solution.** But if $\Delta=0$, then this system has an infinite number of solutions. Hence for non-trivial solution $\Delta=0$.

Example: 23 If the system of linear equations $x+2ay+az=0,\ \ x+3by+bz=0,\ x+4cy+cz=0$ has a non-zero solution, then *a, b, c*

[AIEEE 2003]

(a) Are in A.P. (b) Are in G.P. (c) Are in H.P. (d) Satisfy $a+2b+3c=0$

Solution: (c) System of linear equations has a non-zero solution, then $\begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix}=0$

Applying $C_2 \rightarrow C_2-2C_3$; $\begin{vmatrix} 1 & 0 & a \\ 1 & b & b \\ 1 & 2c & c \end{vmatrix}=0$

Applying $R_3 \rightarrow R_3-R_2$ and $R_2 \rightarrow R_2-R_1$

$\begin{vmatrix} 1 & 0 & a \\ 0 & b & b-a \\ 0 & 2c-b & c-b \end{vmatrix}=0 \Rightarrow b(c-b)-(b-a)(2c-b)=0$. On simplification $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$; $\therefore$ *a, b, c* are in H.P.

Example: 24 If the system of equations $x+ay=0,\ az+y=0$ and $ax+z=0$ has infinite solutions, then the value of *a* is

[IIT Screening 2003]

(a) –1 (b) 1 (c) 0 (d) No real values

Solution: (a) $\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1 + a(a)^2 = 0 \Rightarrow a^3 = -1 \Rightarrow a = -1$

Example: 25 If the system of equations $ax + y + z = 0,\ x + by + z = 0$ and $x + y + cz = 0$, where $a, b, c \neq 1$ has a non-trivial solution, then the value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is

(a) -1 (b) 0 (c) 1 (d) None of these

Solution: (c) As the system of the equations has a non-trivial solution $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$\begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0 \Rightarrow a(b-1)(c-1) - 1(1-a)(c-1) - 1(1-a)(b-1) = 0$$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0 \Rightarrow \frac{1}{1-a} - 1 + \frac{1}{1-b} + \frac{1}{1-c} = 0 \Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Example: 26 If the system of equations $x + 2y - 3z = 1,\ (k+3)z = 3,\ (2k+1)x + z = 0$ is inconsistent, then the value of k is

[Roorkee 2000]

(a) -3 (b) $\frac{1}{2}$ (c) 0 (d) 2

Solution: (a) For the equations to be inconsistent $D = 0$

$\therefore\ D = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & k+3 \\ 2k+1 & 0 & 1 \end{vmatrix} = 0 \Rightarrow k = -3$ and $D_1 = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \neq 0$, Hence system is inconsistent for $k = -3$.

Example: 27. The equations $x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + mz = n$ give infinite number of values of the triplet (x, y, z) if []

(a) $m = 3,\ n \in R$ (b) $m = 3,\ n \neq 10$ (c) $m = 3,\ n = 10$ (d) None of these

Solution: (c) Each of the first three options contains $m = 3$. When $m = 3$, the last two equations become $x + 2y + 3z = 10$ and $x + 2y + 3z = n$.

Obviously, when $n = 10$ these equations become the same. So we are left with only two independent equations to find the values of the three unknowns.

Consequently, there will be infinite solutions.

Example: 28 The value of λ for which the system of equations $2x - y - z = 12$, $x - 2y + z = -4,\ x + y + \lambda z = 4$ has no solution is

[IIT Screening 2004]

(a) 3 (b) -3 (c) 2 (d) -2

Solution: (d) $D = \begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = -3\lambda - 6$. For no solution the necessary condition is $-3\lambda - 6 = 0 \Rightarrow \lambda = -2$. It can be see that for $\lambda = -2$, there is no solution for the given system of equations.

8.1.10 Application of Determinants in Co-ordinate Geometry

(1) Area of triangle whose vertices are $(x_r, y_r);\ r = 1, 2, 3$ is

$$\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

(2) If $a_r x + b_r y + c_r = 0$, $(r = 1, 2, 3)$ are the sides of a triangle, then the area of the triangle is given by

$\Delta = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$, where $C_1 = a_2b_3 - a_3b_2$, $C_2 = a_3b_1 - a_1b_3$, $C_3 = a_1b_2 - a_2b_1$ are the cofactors of the elements c_1, c_2, c_3 respectively in the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

(3) The equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

(4) If three lines $a_r x + b_r y + c_r = 0$; $(r = 1, 2, 3)$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(5) If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines then

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(6) The equation of circle through three non-collinear points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ is

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Example: 29 The three lines $ax + by + c = 0$, $bx + cy + a = 0$, $cx + ay + b = 0$ are concurrent only when

(a) $a + b + c = 0$
(b) $a^2 + b^2 + c^2 = ab + bc + ca$
(c) $a^3 + b^3 + c^3 = ab + bc + ca$
(d) None of these

Solution: (a, b)Three lines are concurrent if $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ or, $3abc - a^3 - b^3 - c^3 = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$

Also, $a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow (a + b + c) = 0$ or $a^2 + b^2 + c^2 = ab + bc + ca$.

8.1.11 Some Special Determinants

(1) **Symmetric determinant :** A determinant is called symmetric determinant if for its every element $a_{ij} = a_{ji} \forall\, i, j$ *e.g.*, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

(2) **Skew-symmetric determinant :** A determinant is called skew symmetric determinant if for its every element $a_{ij} = -a_{ji} \forall\, i, j$ *e.g.*, $\begin{vmatrix} 0 & 3 & -1 \\ -3 & 0 & 5 \\ 1 & -5 & 0 \end{vmatrix}$

Note : ❑ Every diagonal element of a skew symmetric determinant is always zero.

❑ The value of a skew symmetric determinant of even order is always a perfect square and that of odd order is always zero.

For example (i) $\begin{vmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{vmatrix} = -c(0-ab) - b(ac-0) = abc - abc = 0$

(ii) $\begin{vmatrix} 0 & a \\ -a & 0 \end{vmatrix} = 0 + a^2 = a^2$ (Perfect square) (iii) $\begin{vmatrix} 0 & a-b & e-f \\ b-a & 0 & l-m \\ f-e & m-l & 0 \end{vmatrix} = 0$

(3) **Cyclic order :** If elements of the rows (or columns) are in cyclic order.

i.e. (i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(iii) $\begin{vmatrix} a & bc & abc \\ b & ca & abc \\ c & ab & abc \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$

(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$ (v) $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

Note : ❑ These results direct applicable in lengthy questions (As behavior of standard results)

Example: 30 $\Delta = \begin{vmatrix} 0 & i-100 & i-500 \\ 100-i & 0 & 1000-i \\ 500-i & i-1000 & 0 \end{vmatrix}$ is equal to

(a) 100 (b) 500 (c) 1000 (d) 0

Solution: (d) This determinant of skew symmetric of odd order, hence is equal to 0.

Example: 31 $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} =$ **[Pb. CET 1997; DCE 2002]**

(a) $a^2 + b^2 + c^2$ (b) $(a+b)(b+c)(c+a)$ **(c)** $(a-b)(b-c)(c-a)$ (d) None of these

Solution: (c) $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix};$$

Applying $R_1 \to R_1 - R_2$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & (a-c) \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(a-c)\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(a-c)(-1) = (a-b)(b-c)(c-a)$$

Example: 32 If $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$ Where a, b, c are all different, then the determinant $\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix}$ vanishes when

(a) $a+b+c=0$ (b) $x = \frac{1}{3}(a+b+c)$ (c) $x = \frac{1}{2}(a+b+c)$ (d) $x = a+b+c$

Solution: (b) $\because \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$ ...... (i)

Now, $\begin{vmatrix} 1 & 1 & 1 \\ (x-a)^2 & (x-b)^2 & (x-c)^2 \\ (x-b)(x-c) & (x-c)(x-a) & (x-a)(x-b) \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{(x-a)(x-b)(x-c)}\begin{vmatrix} (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ (x-a)(x-b)(x-c) & (x-a)(x-b)(x-c) & (x-a)(x-b)(x-c) \end{vmatrix} = 0$$

Applying $C_1 \to C_1(x-a)$, $C_2 \to C_2(x-b)$, $C_3 \to C_3(x-c)$

$$\begin{vmatrix} (x-a) & (x-b) & (x-c) \\ (x-a)^3 & (x-b)^3 & (x-c)^3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$[(x-a)-(x-b)][(x-b)-(x-c)][(x-c)-(x-a)](x-a+x-b+x-c) = 0$

$(b-a)(c-b)(a-c)[3x-(a+b+c)] = 0$ or $x = \frac{1}{3}(a+b+c)$ $[\because a \neq b \neq c]$

Example: 33 If α, β and γ are the roots of the equations $x^3 + px + q = 0$ then value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is **[AMU 1990]**

(a) p (b) q (c) $p^2 - 2q$ (d) 0

Solution: (d) Since α, β, γ are the roots of $x^3 + px + q = 0$, $\therefore \alpha + \beta + \gamma = 0$

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, We get, $\begin{vmatrix} \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$

Example: 34 If $\Delta = \begin{vmatrix} x+1 & x^2+2 & x(x+1) \\ x(x+1) & x+1 & x(x^2+2) \\ x^2+2 & x(x+1) & x+1 \end{vmatrix} = p_0x^6 + p_1x^5 + p_2x^4 + p_3x^3 + p_4x^2 + p_5x + p_6$,then $(p_5,\ p_6) =$

(a) (-3, -9) (b) (-5, -9) (c) (-3, -5) (d) (3, -9)

Solution: (b) Putting $x = 0$ in both sides, we get, $\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{vmatrix} = p_6 \Rightarrow p_6 = 9$ by expansion.

p_5 is the coefficient of x or constant term in the differentiation of determinant.

Differentiate both sides,

$$\begin{vmatrix} 1 & 2x & 2x+1 \\ x(x+1) & x+1 & x(x^2+2) \\ x^2+2 & x(x+1) & x+1 \end{vmatrix} + \begin{vmatrix} x+1 & x^2+2 & x(x+1) \\ 2x+1 & 1 & 3x^2+2 \\ x^2+2 & x(x+1) & x+1 \end{vmatrix} + \begin{vmatrix} x+1 & x^2+2 & x(x+1) \\ x(x+1) & x+1 & x(x^2+2) \\ 2x & 2x+1 & 1 \end{vmatrix}$$

$$= 6p_0x^5 + 5p_1x^4 + 4p_2x^3 + 3p_3x^2 + 2p_4x + p_5$$

Putting $x = 0$ both sides, we get $p_5 = -5$; $\therefore\ (p_5,\ p_6) = (-5,\ 9)$.

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